X-641-69-441
PREPRINT

NASA TM X-63732

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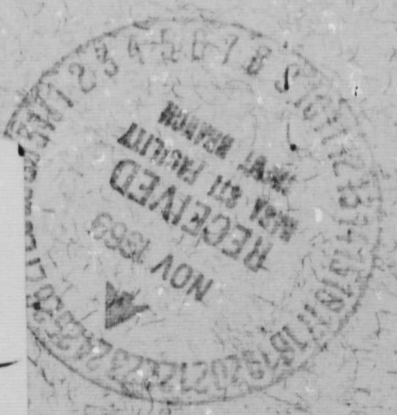
OCTOBER. 1969



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ABSTRACT

Two recent approaches to the statistical theory of turbulence based on the distribution function approach, the "Generalized random phase approximation" [S. F. Edwards, J. Fluid Mech. 18, 239 (1964)] and the "Self-Consistent Field Approximation" [J. R. Herring, Phys. Fluids 8, 2106 (1966)], are described and compared. These procedures are viewed as alternate ways of expanding the flows probability distribution about its own univariate distribution. The "Generalized random phase approximation" models the single-mode-turbulence dynamics by a Fokker-Planck equation, while the self-consistent-field procedure uses a similar, but more general formula. Both procedures have as ingredients, the simultaneous time correlation coefficients and "eddy" relaxation frequencies. Their second order results differ only in the equations determining these eddy relaxation frequencies. The procedures are formally similar to (but simpler than) the direct interaction approach.

The accuracy of these procedures is assessed by comparing their predictions for the energy spectra of some simple model systems with exact numerical results. Results are also presented for moderate Reynolds number turbulence, and these are compared to similar results for the direct interaction approximation.

Finally, we discuss the failure of these theories to predict a proper Kolmogorov inertial range spectrum. The difficulty lies with the Green's function (or relaxation frequency) equations; a suggestion for eliminating this flaw is made.

TURBULENCE FROM THE DISTRIBUTION FUNCTION POINT OF VIEW

I. INTRODUCTION

The statistical theory of turbulence may be concisely formulated in terms of the distribution function of the velocity field $P(v, t)$, defined to be the probability that at time t the velocity field at all spacial points $(r_1, r_2, r_3, \cdots, r_n, \cdots)$ is within a small neighborhood $d\Omega = dv_1\, dv_2 \cdots dv_n \cdots$ of $\left(v(r_1), v(r_2), \cdots v(r_n) \cdots\right)$. A knowledge of P enables one to compute any desired moment of the velocity field. For example, the energy spectrum is found from

$$\left\langle v(r_1)\, v(x_2)\right\rangle \equiv \phi(r_1, r_2, t) = \int d\Omega\, v(r_1)\, v(r_2)\, P(\cdots v \cdots, t) .$$

To simplify the notion, we dispense with the vector indices of $\vec{v}$. We also enumerate the continuum of spatial points as if they were discrete in order to stress the physical notations of the method. The Navier-Stokes equations imply an equation of evolution for P which is a statement of conservation of probability in the phase space $d\Omega = (v_1, v_2, \cdots)$. Abbreviating the Navier-Stokes equations by $\dot{v}(x) = F(v(x))$, the equation for P is

$$\frac{\partial P}{\partial t} = \int \frac{\partial}{\partial r_i}\left(F_i\left(v(r_1), v(r_2), \cdots\right) P(t)\right) dr \qquad (1)$$

The hierarchy of moment equations is recovered from (1) by multiplying it by suitable products of v and integrating over $d\mathcal{V}$.

We discuss here two recent approaches to the statistical theory of turbulence which use Equation (1) as their starting point. These are the generalized random phase approximation (GRD) (S. F. Edwards, 1964) and the self consistent field approximation (SCF) (J. R. Herring, 1965, 1966). The starting point of these procedures differ from that of the more familiar moment approaches in that an approximate solution to (1) for P is first found, and from this solution an equation for the energy spectrum is worked out.

Both these approaches (the S.C.F. and the G.R.P. approximations) treat by different techniques stationary turbulence (driven by some negative viscosities or by random stirring forces) and time dependent turbulence. Stationary turbulence is treated by putting $\partial P/\partial t = 0$, and then seeking an approximate solution for (1). Time dependent turbulence is treated by introducing relaxation operators which represent the effects of $\partial P/\partial t$. The equations for the time dependent case contain integrals over the history of the turbulence, whereas for the stationary case no such complications occur. For the S.C.F. procedure the time dependent results do not reduce to the stationary state case in the limit of large times.

The moment equations obtained from the time dependent S.C.F. method are closely related to the direct interaction (DI) equations of Kraichnan (1959). The main difference between the two methods is that the S.C.F. method involves no nonsimultaneous time moments. It therefore represents a simplification over the latter.

We begin with a discussion of the time dependent S.C.F. method, which is presented as a generalization of the quasi-normal approximation. There follows a comparison of this method with the direct interaction approximation for the case of the decay of an initial turbulent field. The S.C.F. and G.R.P. approximations are then compared for the case of stationary turbulence driven by a random stirring force. Finally we discuss the failure of these methods to yield a Kolmogorov inertial range spectrum; for the S.C.F. method a suggestion for removing this flaw is made.

II. TIME DEPENDENT THEORY

Instead of (1) we use an equivalent Fourier mode distribution $P(x_1, x_2, \cdots; t)$ where x_i denotes collectively the real and imaginary part of the complex Fourier transform of $v(x, t)$,

$$v(k) = \int e^{ik\cdot r}\, v(r, t)\, dr .$$

Then if the equations of motion for x_i are denoted by $\dot{x}_i = F_i(x_1, \cdots x_N)$ the equation for P is,

$$\frac{\partial P}{\partial t} = \sum_i \frac{\partial}{\partial x_i}\left(F_i P\right) . \qquad (2)$$

The explicit form of F is,

$$F_i = -\nu k_i^2 x_i + \sum M_{ijk} x_j x_k .$$

The advantage of the Fourier mode description lies in the fact that for homogeneous turbulence $\langle v(k)\, v(k') \rangle = 0$ unless $k + k' = 0$, whereas $\langle v(x_1)\, v(x_2) \rangle \neq 0$ for any x_1 and x_2. To compute the energy spectrum $\langle |v(k)| \rangle^2$ we only need the single mode distribution function $\mathcal{P}(x_i)$, defined by,

$$\mathcal{P}(x_i, t) = \int dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots P .$$

If the turbulence force were Gaussianly random,

$$P = \prod_i \mathcal{P}(x_i) .$$

This product form holds only for Fourier modes.

We seek a valid approximation for the single mode distribution $\mathcal{P}$. The final equations obtained here heuristically may also be obtained as the leading terms of a formal perturbation series (Herring 1965). However, such a formalism seems of limited value since the perturbation series probably diverges. We start with an equation which is the distribution function equivalent of the

quasi-normal approximation for moments. It is,

$$\left(\frac{\partial}{\partial t} + \Gamma_i^0\right) \Phi_i = \int_0^t dt' \int_{-\infty}^{\infty} dx' \left\langle V(x)\, G(x, x'; t, t')\, V(x') \right\rangle_i \Phi\left(x_i', t'\right) \quad (3)$$

where, $G = G_0$

$$\Gamma_i^0 = -\nu k_i^2 \frac{\partial}{\partial x_i} x_i$$

$$V = \sum_{i,j,k} M_{ijk} \frac{\partial}{\partial x_i} x_j x_k$$

and,

$$G_0 = \exp\left\{(t - t') \sum \Gamma_i^0\right\} \cdot \prod \delta\left(x_i - x_i'\right)$$

$$= \prod \exp\left\{\nu k_i^2 (t - t')\right\} \delta\left(x_i\, e^{\nu k_i^2 (t - t')} - x_i'\right)$$

$$= \prod G_i^0 .$$

In (3), the angular brackets applied to an operator O have the following meaning;

$$\langle O \rangle_i = \int dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots O \prod_{j \neq i} \Phi\left(x_j, t\right)$$

The standard moment quasi-normal approximation follows from (3) by multiplication by x_i^2 and integrating over dx_i. It is,

$$\left(\frac{\partial}{\partial t} + 2\nu k_i^2\right)\psi_i = \sum M_{ijk} \int_0^t g_{ijk}^0(t,t') \left[M_{ijk}\psi_j(t')\psi_k(t') + M_{jik}\psi_i(t')\psi_j(t') + M_{kij}\psi_i(t')\psi_k(t')\right] dt'$$

where,

$$g_{ijk}^0 = \exp\left\{-\nu(t-t')\left(k_i^2 + k_j^2 + k_k^2\right)\right\}.$$

Equation (3) is very close to the Fokker-Planck equation; to obtain the latter, one has only to update the time arguments of the P's from t' to t. This updating would be justified if the time scale of G_0 is infinitely rapid compared to that of P. Such a modification would prevent the ψ_i spectrum from evolving negative values as has been found by (Ogura (1963)). Since, however the time scale of G_0 is nominally the same as that of P, there remains the question of how to modify (3) so as to obtain a valid approximation.

The trouble with (3) is that at large Reynolds numbers it does not provide a way whereby the initial form of P ceases to influence its behavior at large times. This lack of 'memory loss' is reflected in the fact that the form of G_0

corresponds to phase space trajectories which remain statistically sharp during the entire interval (t - t′). On the other hand, the actual trajectories have complicated meanderings caused by the turbulence forces. An ensemble average over such meanderings would produce a smoothed G, which for fixed x_n diffuses to zero as (t - t′) increases. The proposal for improving (3) is to include memory loss effects by replacing G_0 by the exact G.

There remains the task of determining G. This is done in the approximation that the interactions are individually feasible, so that G may be approximated by its product-mode-form;

$$G \approx \prod_i G_i , \qquad G_i = \int dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots G . \tag{5}$$

An approximation for G_i may be found by appreciating the fact that it is a single mode distribution function which is statistically sharp at t = t′: Hence, G_i satisfies (3) with G replacing G_0:

$$\left(\frac{\partial}{\partial t} + \mathfrak{L}_i^0\right) G_i (t, t') = \delta\left(x_i - x_i'\right) \delta(t - t')$$

$$+ \int_0^t dt'' \left\langle VG(t, t'') V \right\rangle_i G_i (t'', t') . \tag{6}$$

The moment ψ_i is then determined from the modified form of (3) and (6). Defining,

$$g_i(t, t') \equiv \int_{-\infty}^{\infty} x_i \, dx_i \int_{-\infty}^{\infty} dx_i' \, G(x_i, x_i') \frac{\partial}{\partial x_i'} U(x_i', t') \quad ,$$

there follows from (5) and (6)

$$\left(\frac{\partial}{\partial t} + \nu k_i^2\right) g_i(t, t') = \delta(t - t') + \int_{t'}^{t} dt'' M_{ijk} \left[M_{jik} \psi_i(t') g_k(t, t'') + M_{kij} \psi_k(t') g_i(t, t'') \right] g_i(t'', t') \quad . \quad (7)$$

The equation for ψ is found to be identical to (4) with g as given by (7) replacing g_0.

Equations (4) and (7) may be formally obtained from the D.I. equations by discarding the $\psi(t, t')$ equations, replacing them with

$$\psi(t, t') \rightarrow g(t, t') \psi(t', t') \, .$$

III. COMPARISON OF THE S.C.F. AND D.I. APPROXIMATIONS

The S.C.F. approximation is similar to but simpler than the D.I. approximation. In an initial value problem, the amount of numerical labor in a S.C.F. computation is less than half that necessary for the equivalent D.I. computation.

It is therefore of interest to inquire as to the relative accuracy of the two methods. Such a comparison has already been made for some simple three and five mode inviscid models (Herring, 1966). There it was concluded that the S.C.F. procedure had the same qualitative behavior as the D.I. approximation. Typically, both procedures predicted the asymptotic state to be approached by way of spurious damped oscillations. The size of the excursions was slightly larger for the S.C.F. than the D.I. approximation, but this type of error tended to decrease with increasing number of modes.

Results for the two methods for the complete Navier-Stokes equations is presented in Figures 1, 2, and 3. These curves are for the decay of isotropic turbulence, whose initial energy spectrum is,

$$E(k, 0) = 16\sqrt{\frac{2}{\pi}}\,\frac{k^4}{k_0^5}\exp\left(-2\left(\frac{k}{k_0}\right)^2\right), \qquad k_0 = 4.75683 .$$

This spectrum was used by Ogura in his study of the quasi-normal approximation and also by Kraichnan (1964). Our results for the D.I. method differ a bit from Kraichnan's because of slightly different numerical procedures in the two cases. The main difference is that in the present calculation, linear k-steps are used in the wave number discretization instead of logarithmic steps. Also, we have averaged the geometrical $A(k, k', k'')$ and $B(k, k', k'')$ coefficients over the discretizing k-volume instead of using their midpoint value, as done by Kraichnan.

The evolved energy spectra, shown in Figure 1 (for t .76) indicates the two methods to be in close agreement over the entire energy containing range of k. The D.I. approach gives a slightly higher transfer rate out of the energy containing region, and hence slightly more energy at larger k. This fact is also brought out in Figure 2, which compares predictions for the skewness factor,

$$S(t) = -\left\langle\left(\frac{\partial u}{\partial x}\right)^3\right\rangle \Big/ \left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle^{3/2} .$$

for the two methods. Figure (3) compares the D.I. value of $\psi_k(t \mid t')$ with the S.C.F. value of $\psi_k(t' \mid t')\, g_k(t \mid t')$, for k near the peak of the energy curve, and for $t = 1.76$. The agreement in this case is not so close; the S.C.F. method predicts a quicker memory loss of the initial state. The g_k for the two methods are graphically the same for all $k \leq k_{max} = 40$.

The procedures were also compared for the other spectra investigated in Kraichnan 1964 article (spectra C and D). Agreement in these cases was better than could be distinguished by present graphical accuracy.

IV. STATIONARY TURBULENCE

Stationary turbulence is treatable by the method of the preceding section as the asymptotic limit of a flow field driven by either a stationary random force or by some negative viscosities, which simulate production of turbulence by shear instabilities. In the $t \to \infty$ limit, all statistical quantities in (4) and (7) become functions of $(t - t')$ alone. The resulting equations may be solved by

iteration. An alternate approach to the stationary problem is to first discard P in (1), and then develop an approximation to solve the time independent P equation. The second alternative is attractive in that the ensuing analysis involves no integrals over the flow's past. The final equations are thus simpler, and easier to solve than the t $\to \infty$ limit of (4) and (7).

This second approach has been developed by two related techniques in the papers of S. F. Edwards (1964) and the present author J. R. Herring (1965). These stationary methods use, instead of Green's functions relaxation operators, which are related to time integrals of the former. These operators determine the single mode distribution function P_n through an equation $\mathcal{L}_n P_n = 0$, which is analogous to a stationary state Boltzmann equation.

The form of $\mathcal{L}_n$ is determined by a perturbation procedure, whose leading term for P is ΠP_n. The difference between the S.C.F. and G.R.P. method is that the latter explicitly takes a Fokker-Planck form for $\mathcal{L}_n$ whereas the former attempts to determine its form self-consistently using information from the perturbation series.

Both these methods have a degree of arbitrariness traceable to the fact that the perturbation methods use as a smallness parameter the Reynolds number, R, to order the terms of the series in presumed order of decreasing magnitude. The ordering is plausable at small R, but can be correct at large R only if the turbulence force in its effect on P is very nearly Gaussian. At large R consecutive terms of the series are nominally of the same size so that at best the series is asymptotic.

The stationary S.C.F. approximation may be obtained as a modification of the time dependent method of Section II by putting $G(t) = \exp(-\eta_n t)$ then integrating (4) and (7) over t from 0 to ∞, with the result

$$\nu_n \psi_n = \sum \left(2M^2_{npq} \phi_p \phi_q + 4M_{npq} M_{pnq} \phi_n \phi_q\right) \frac{1}{\eta_n + \eta_p + \eta_q} \tag{9}$$

$$\eta_n = \nu_n - 4 \sum M_{npq} M_{pnq} \phi_q \frac{1}{\eta_p + \eta_q} \cdot \tag{10}$$

The G.R.P. approximation yields results identical to (9) and (10) except that the denominator of (10) is replaced by $(\eta_n + \eta_m + \eta_L)$. In both methods η_n is the smallest non-zero eigenvalue of $-\mathfrak{L}_n$. In order to maintain the assumed stationarity, (10) should either have some negative ν_n's or a term representing a driving force should be added to it.

Equations (9) and (10) are not identical to the stationary state form of (4) and (7). It is not apparent which approximation is better. Equations (4) and (7) appear more easily interpretable in physical terms. On the other hand, they may erroneously treat historical effects of which the stationary method makes no mention. In this connection it is interesting to recall that the quasi-normal approximation (which may be regarded as the zeroth order term of the present method) is known to fail for large Reynolds numbers in its time dependent form, whereas its stationary form (obtained from (9) and (10) by replacing η_n by ν_n) has no obvious non-physical behavior.

A comparison of the stationary methods is given in Figure (4). We include in this comparison the stationary quasi-normal approximation. Here we show the energy spectrum for the case in which the turbulence is maintained by a white noise stirring force, whose spectral shape is $k^2 E(k, 0)$, where $E(k, 0)$ is given by (8). Values of the integral scale Reynolds number, R_L, are given in the figure. The S.C.F. and G.R.P. approximations give comparable results. The G.R.P. method has slightly more energy transferred to higher wave numbers, which is consistent with the fact that it has a smaller eddy viscosity coefficient. The quasi-normal approximation, which has no eddy viscosity has a much larger transfer rate than either of the other methods.

V. CONCLUDING COMMENTS

Finally we must point out a defect these approximations share with the direct interaction method; that is, their failure to correctly predict the Kolgomorov spectrum at large Reynolds numbers. As is well known (Kraichnan, 1964) all these theories predict a - 3/2 instead of a - 5/3 inertial range. The trouble arises from an incorrect treatment of the interactions of very large and small wave numbers. Suppose a bit of energy δE is added to the extremely low wave number end of the spectrum. Since this addition represents an almost uniform translation velocity added randomly to members of the ensemble, it should not modify the large wave number energy spectrum. This follows from the Galilean covariance of the Navier-Stokes equations. Yet such an addition does modify the predictions

made by (9) and (10), because (10) now has an extra term MMδE/$(\eta + \eta)$ added to the large k-equations. The energy Equation (9) is changed only because of the η_n change; if η_n is unmodified the δE terms in (9) cancel.

These difficulties result from an inappropriate mixture of ingredients to treat large Reynolds numbers flows. The theories use amplitude Green's functions (or relaxation frequencies) to evolve the energy spectrum-not an amplitude but a second order moment. There is nothing obviously wrong with the amplitude Greens function equations; in fact they predict reasonable results. It is only their use in energy equations that is here questioned.

The difficulty may be avoided by using a theory whose basic ingredients contain intensity Green's functions in place of the amplitude Green's functions. At present, we are investigating a theory produced by modifying the S.C.F. procedure along the following lines. First, return to x-space formalism where it is easier to keep track of Galilean invariance. Then instead of using a perturbation Green's function use the exact one for the simple case of zero viscosity. In this case the exact Greens function simply evolves the trajectories of particles convected with the fluid. Next, make the diffusion approximation in phase space, to get an equation analogous to (3). Then approximate the Green's function equation with a soluble one. This turns out to be closer to an intensity equation than an amplitude equation. Finally, reintroduce viscous effects additively.

An alternate approach is to abandon a deductive determination of η_n entirely, using instead some principle for its determination. This approach has been recently pursued by Edwards and McComb (1969), who adapt an information

theoretic approach for choosing those η_n which extremalize the entropy

$$S = \int d\Omega P \ln P$$

consistent with the constraint of conservation of energy by nonlinear interactions.

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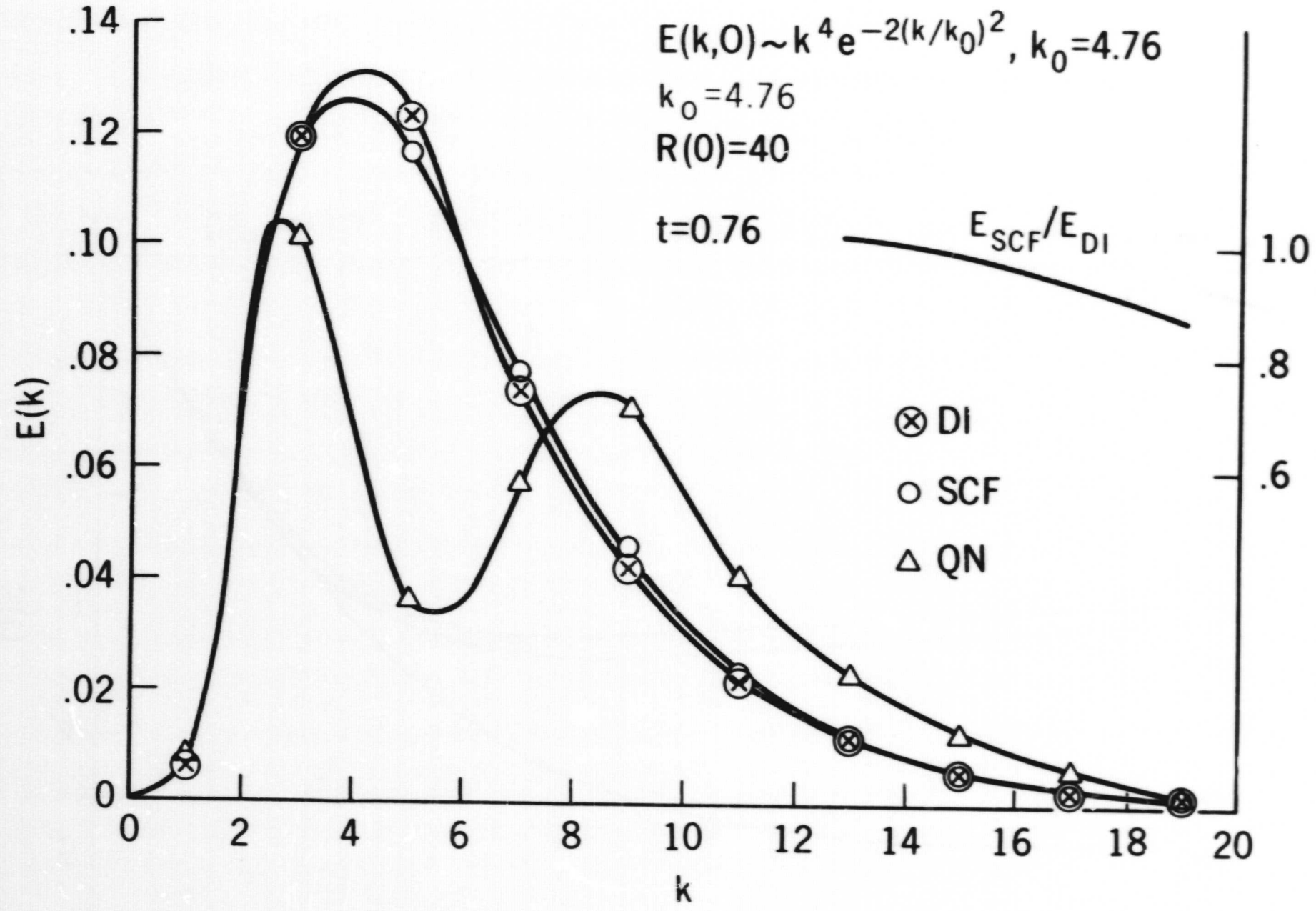

Figure 1. Evolution of energy spectrum E(k, t) according to the direct interaction approximation (⊕); the self-consistent field (⊙); and the quasi-normal (△) approximation. The initial energy spectrum is $E(k, 0) \sim k^4 \exp\left(-2(k/k_0)^2\right)$, $k_0 = 4.76$.

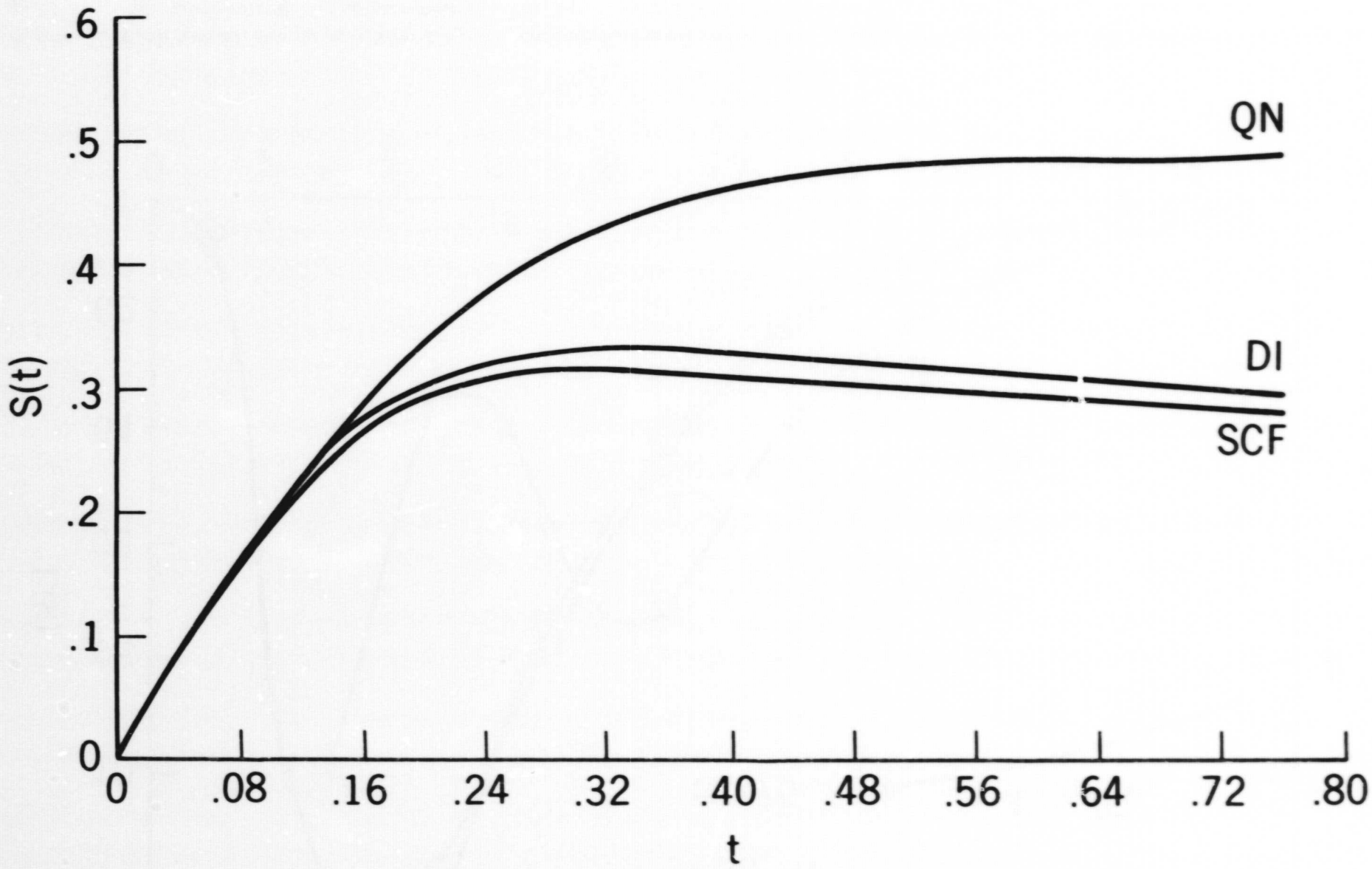

Figure 2. Evolution of skewness factor S(t) according to approximations described in Figure 1

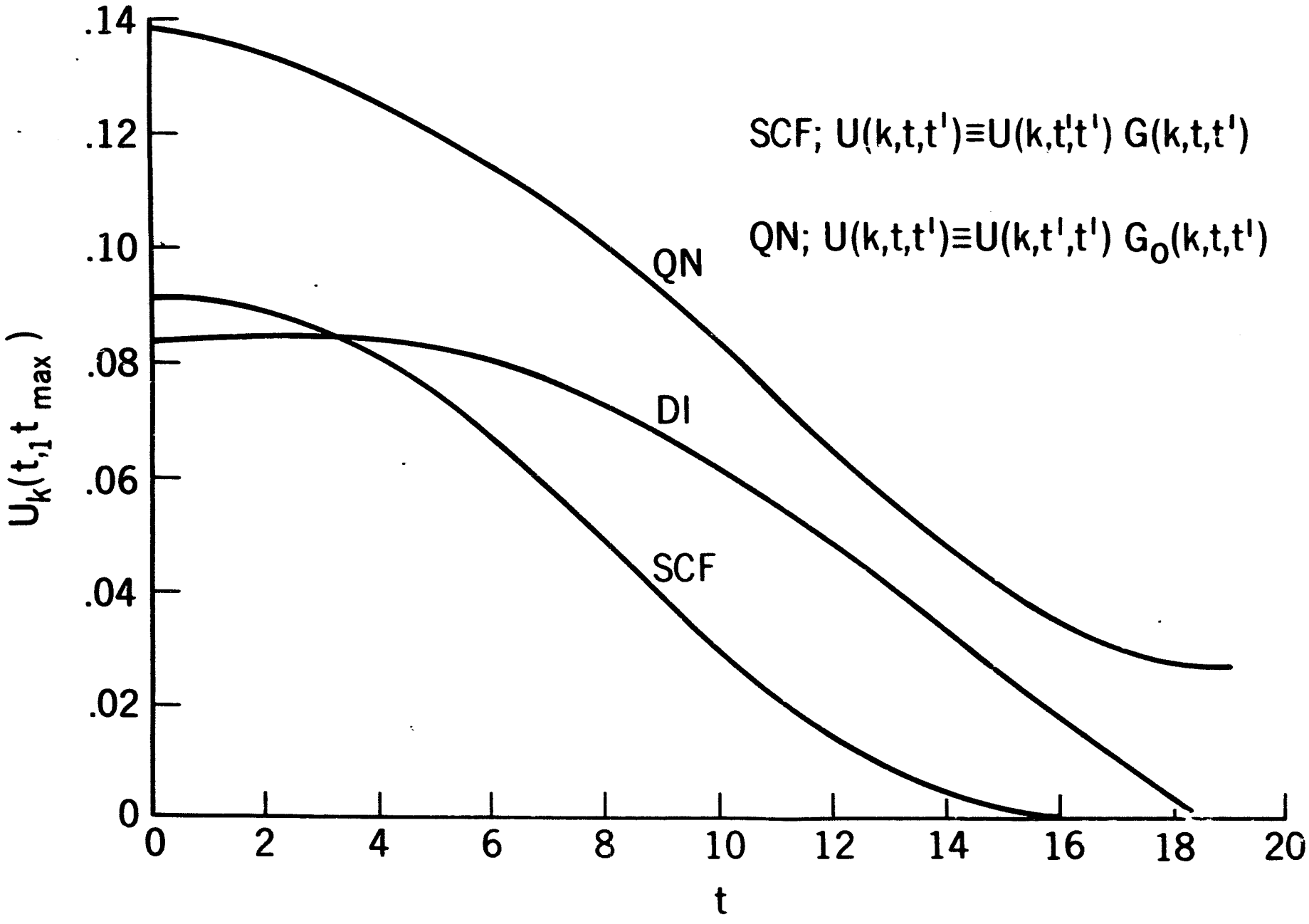

Figure 3. Non-simultaneous correlation function U(t, t') according to the three approximations of Figure 1, $k = 2k_0$.

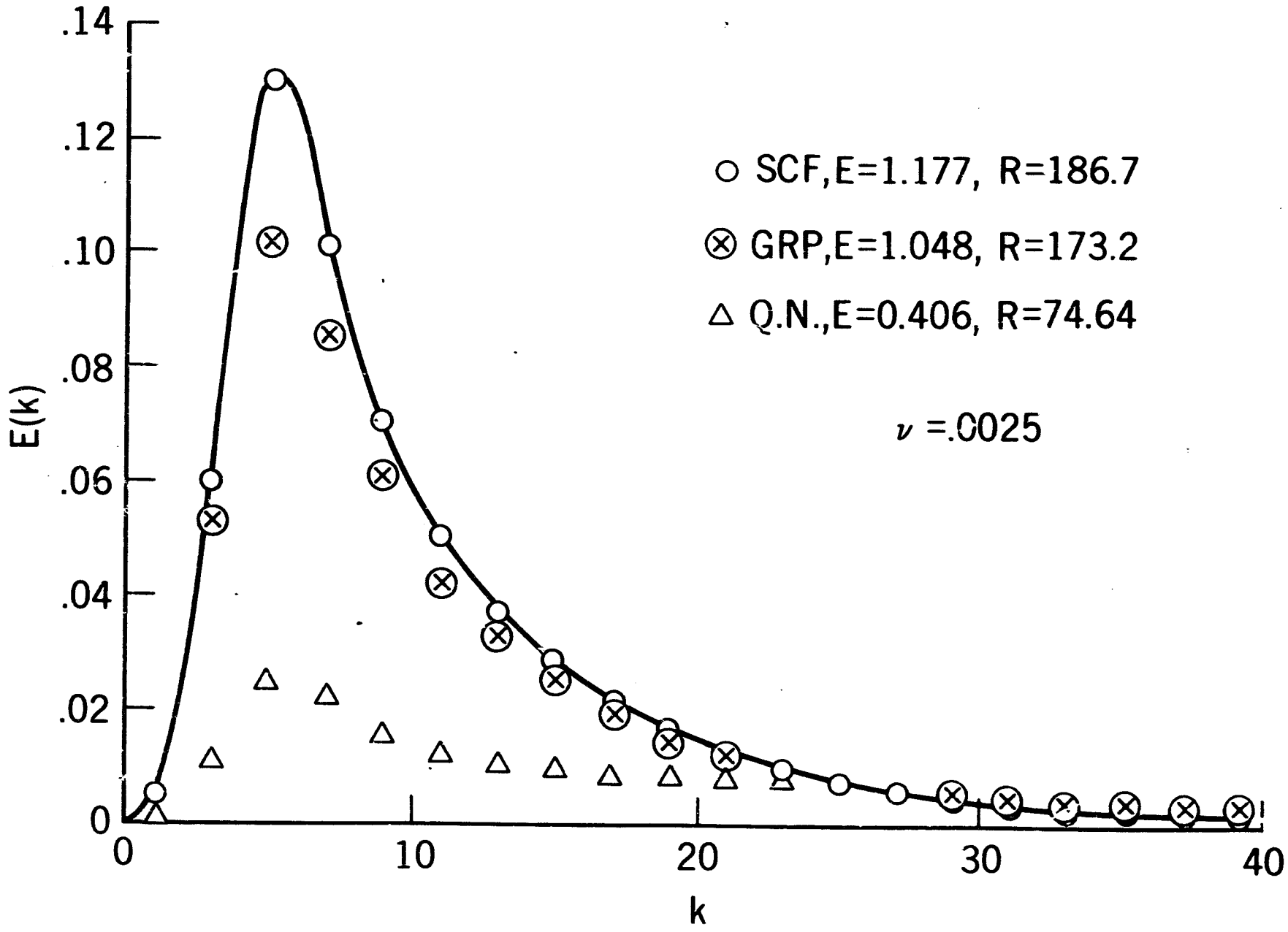

Figure 4. Stationary turbulence maintained by a white noise stirring force whose spectrum is $\sim k^2 \exp\left(-2(k\ k_0)^2\right)$ according to the self-consistent field (⊙); the generalized random phase approximation (⊕); and the quasi-normal approximation (△).